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Introduction to nonlinear discrete systems: theory and modelling

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Abstract

An analysis of discrete systems is important for understanding of various physical processes, such as excitations in crystal lattices and molecular chains, the light propagation in waveguide arrays, and the dynamics of Bose-condensate droplets. In basic physical courses, usually the linear properties of discrete systems are studied. In this paper we propose a pedagogical introduction to the theory of nonlinear distributed systems. The main ideas and methods are illustrated using a universal model for different physical applications, the discrete nonlinear Schrödinger (DNLS) equation. We consider solutions of the DNLS equation and analyse their linear stability. The notions of nonlinear plane waves, modulational instability, discrete solitons and the anti-continuum limit are introduced and thoroughly discussed. A Mathematica program is provided for better comprehension of results and further exploration. Also, a few problems, extending the topic of the paper, for independent solution are given.

Supplementary material for this article is available online

Keywords: discrete systems, lattices, arrays, chains, discrete nonlinear Schrodinger equation, discrete solitons, Mathematica code

(Some figures may appear in colour only in the online journal)

1. Introduction

A discrete system is a system that consists of several (or an infinite number of) well-separated points (sites). Each site is characterised by some variables, so that at a given time these variables specify a state of the system. A change of variables on each site may depend on values on other sites.
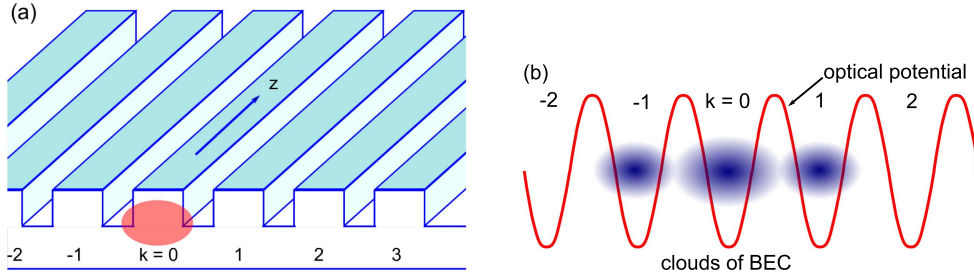
Figure 1. Examples of systems, described by the DNLS equation. (a) An array of optical rib waveguides. Light in a waveguide (a spot at $k = 0$) is coupled with nearest-neighbour waveguides. (b) Clouds of Bose–Einstein condensate, distributed in minima ($k = -1, 0$ and 1) of the periodic optical potential.

Many important physical systems, such as crystal lattices, atomic chains, polymers, arrays of resonators and waveguides, electrical transmission lines, and spin lattices are essentially discrete [1–6]. A recent advance in this field is summarised in books and research papers, that are written mainly for specialists. The purpose of this paper is to give an accessible introduction to the subject, and to illustrate the analytical methods used.

Different aspects of discrete systems are discussed in educational literature (see e.g. [7–11]) to enhance the corresponding university courses. However, mainly linear properties are studied in such courses. In this work, the basic ideas of the nonlinear dynamics are presented. We demonstrate main steps for analysis of discrete systems, using a well-studied model, the discrete nonlinear Schrödinger (DNLS) equation [1–6]. We analyse properties of plane waves and localised modes. We also provide a script in Mathematica that helps to do the analysis. A student can test these methods on the DNLS equation, and then extend and modify them to study more complex systems. Our paper can also be useful for lecturers, who want to include this topic to courses on nonlinear physics.

The DNLS equation is written in the following form:

$$i\frac{du_k}{dt} + \beta_0 u_k + \beta(u_{k+1} + u_{k-1}) + \gamma |u_k|^2 u_k = 0, \tag{1}$$

where u_k is the field value at kth point, k is an integer number, $\beta > 0$ characterises a coupling with neighbour sites, and γ is the nonlinearity parameter. In this section, we give examples of different physical systems described by equation (1).

Let us consider an array of optical waveguides, see figure 1(a). The field variable u_k corresponds to the electric field E in kth waveguide. Waveguides are situated close to each other, so that light in one waveguide can tunnel into nearest-neighbour waveguides. This effect induces an overlap of modes and interaction of fields in the adjacent waveguides (see the third term in equation (1)). It is assumed that the refractive index n of the material is intensity-dependent, $n = n_0 + n_2|E|^2$. Such a dependence of n results in linear (the second) term and nonlinear (the fourth) term in equation (1). Therefore, equation (1) describes light propagation in a waveguide array. The equation can be derived from the Maxwell equations in the limit of weakly coupled modes [1, 4, 6]. Parameter β_0 is the linear propagation constant, β is the coupling coefficient, and $\gamma \sim n_2$ characterises the Kerr nonlinearity. The propagation distance z plays here the role of time. A study of beam dynamics in arrays can be useful for light routing in photonic circuits, for beam steering and switching [1, 4, 6]. Waveguide arrays can serve as a universal testbed for analysis of phenomena from solid state physics, such as Anderson localisation and Bloch oscillations.

Now, we consider a gas of ultra-cold (e.g. rubidium) atoms at temperature $\sim 10^{-7}$ K in an optical lattice, see figure 1(b). Because of such a low temperature, all atoms are in the ground state, and they form a new state of matter, a Bose–Einstein condensate (BEC) [12]. An optical lattice, which is a standing wave created due to interference of laser beams, forms a periodic potential for a BEC. Let us analyse a BEC distributed in the minima of the potential. A relative number of atoms in the kth minimum are described by $|u_k|^2$. Then, assuming that particles on different minima interact weakly due to tunnelling, and taking into account the two-body scattering effect, one can arrive to equation (1). More rigorous derivation of equation (1) from the Gross–Pitaevsky equation for a BEC in a periodic potential can be found in [13]. A BEC in optical lattices, as an ensemble of coherent objects, is actively studied from a fundamental point of view. A set of interacting condensate droplets can be used in experiments on matter wave interference, and precise measurements [12].

The DNLS equation (1) describes also other systems, such as Josephson junction arrays, layered magnetic systems, organic molecules and DNA, see [3, 5, 6]. Since the DNLS equation has different applications, in this work we treat it in general terms, referring to u_k as the field variable and to t as time. Now, when the physical importance of equation (1) is justified, let us consider its main properties.

Equation (1) has two conserved quantities:

$$P = \sum_{k=1}^{N} |u_k|^2, \quad H = \sum_{k=1}^{N} \left(\beta_0 |u_k|^2 + \beta |u_k - u_{k-1}|^2 + \frac{\gamma}{2} |u_k|^4 \right). \tag{2}$$

In the context of waveguide arrays, P is the total power, while H is the Hamiltonian. The second term in equation (1) can be eliminated by using transformation $u_k(t) \rightarrow u_k(t) \exp(\mathrm{i}\beta_0 t)$. In the rest of the paper, we assume $\beta_0 = 0$.

When $\gamma = 0$, an excitation, initially localised on a single site, spreads across the array due to coupling. Nonlinearity ($\gamma \neq 0$) can support localised states, in which the energy is locked mainly in few sites. These localised states are called discrete solitons (DSs). There are two basic types of solitons, namely, bright solitons and dark solitons. The names come from applications in optics. A bright DS corresponds to a distribution that vanishes far from the mode centre. In such a mode is observed a set of bright spots located in few waveguides. A dark DS corresponds to a dip on a constant background. We focus our attention on bright DSs.

Plane waves and DSs are considered as fundamental modes of nonlinear discrete systems. Any initial field distribution with finite P ends up typically in a set of spreading waves and DSs [2, 5]. Therefore, in order to study the dynamics of discrete systems, one needs to understand properties of these excitations.

2. Analysis of waves in discrete systems

2.1. Plane waves

For discrete systems, it is instructive to start the analysis from a plane wave solution. We look for a solution in the following form:

$$u_k(t) = a \exp[\mathrm{i}(qk - \omega t)], \tag{3}$$

where a, q, and ω are the amplitude, the wave number and the frequency of the plane wave. A substitution of equation (3) into (1) results in the following dispersion relation:

$$\omega = -(2\beta \cos q + \gamma a^2). \tag{4}$$

This is an important characteristics of plane waves. The physical meaning of the dispersion relation is simple. If at $t = 0$, one prepares a profile in a form of equation (3) with given q, then the real and imaginary parts oscillate in time with frequency ω. Alternatively, exciting in time a single site with frequency ω, one generates a wave with wave number q defined by equation (4).

First, we consider the dispersion relation of the linear system, $\gamma = 0$ (or $a \to 0$). The first derivative $d\omega/dq$ defines the group velocity, while the second derivative $d^2\omega/dq^2$ is the wave dispersion. Since, in general, $d^2\omega/dq^2 \neq 0$, we conclude that the inter-site coupling induces effective dispersion. This dispersion is responsible for a spreading of initially localised distribution. Another property, worth noticing, is that dispersion can change its sign, depending on the value of q. We return to this property in the discussion of localised waves.

In presence of nonlinearity, $\gamma \neq 0$, the dispersion relation shifts up or down depending on the sign of γ. The dependence of the wave frequency on the amplitude is typical for nonlinear systems.

For $q = 0$ ($q = \pi$), amplitudes ae^{iqk} in nearest-neighbour sites have the same (opposite) sign. The corresponding distribution is called unstaggered (staggered).

The next step of the analysis is to study the dynamics of small perturbations of the plane wave. A growth of small perturbations is a manifestation of modulational instability (MI). Usually, the result of MI is a structure with well-separated localised waves that can be associated with DSs.

In order to find unstable wave parameters, we represent the solution in the following form:

$$u_k(t) = [a + b_k(t)]\exp[i(qk - \omega t)], \tag{5}$$

where $b_k(t)$ is a complex amplitude of small modulations of the plane wave. Substituting equation (5) into (1), and taking only first-order terms on b_k, we get the following linear equation for perturbations:

$$i\frac{db_k}{dt} + \omega b_k + \beta(b_{k-1}e^{-iq} + b_{k+1}e^{iq}) + \gamma(2a^2 b_k + a^2 b_k^*) = 0. \tag{6}$$

Separating the real and imaginary parts of b_k, and representing them $\sim\exp[i(Qn - \Omega t)]$, we obtain a linear algebraic system. The compatibility condition of this system gives the dispersion relation of perturbations, see [5]:

$$(\Omega - 2\beta \sin Q \sin q)^2 = 8\beta \sin^2(Q/2)\cos q \, [2\beta \sin^2(Q/2)\cos q - \gamma a^2]. \tag{7}$$

When the right-hand side of equation (7) is negative, Ω is complex, and plane wave (3) is unstable.

Figure 2 shows the dynamics of a plane wave, modulated initially by noise, $u_k(0) = 1 + \epsilon r_k$, where r_k is a random number with uniform distribution in $[-1, 1]$, and $\epsilon = 10^{-3}$. Numerical simulations, as in figure 2, can be obtained by commands, similar to those in steps 1 and 4 (MC.1 and MC.4[3]) of the Mathematica code in the appendix; see also Problem 1 in section 3.

Since noise has waves with arbitrary Q, some of the modulation wavenumbers are in the unstable region. As a result of instability, a plane wave breaks into a set of soliton-like pulses. Though the amplitudes of pulses in figure 2 are varied, the shapes and positions remain basically the same. Therefore, MI indicates a presence of DSs in the system. We analyse *stationary* DSs in the next section.

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[3] In the text, for brevity we use notation 'MC.*n*' that refers to *n*th step of the Mathematica code in appendix.
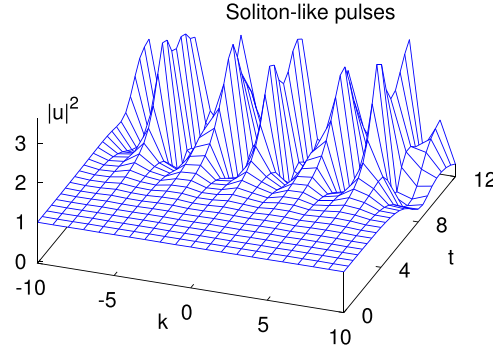
Soliton-like pulses

Figure 2. The dynamics of a plane wave with random modulations. The plane wave breaks up due to MI into a set of soliton-like pulses.

2.2. Discrete bright solitons

There is no exact DS solutions of the DNLS equation. One can construct approximate solutions for different limiting cases. One limit is the case when the soliton width is much larger than the distance between sites. In other words, the variation of u_n from one site to other is small. Then equation (1) can be approximated by the continuous NLS equation:

$$\mathrm{i}\frac{\partial w}{\partial t} + 2\beta h^2 \frac{\partial^2 w}{\partial x^2} + \gamma |w|^2 w = 0, \tag{8}$$

where $w(x_k, t) \approx u_k(t)\exp(-2\beta t)$, and h is the distance between neighbour sites. The NLS equation is the completely integrable model [2] that has a vast number of exact solutions. We can use the soliton solution of the NLS equation in order to construct an approximate expression for the DS:

$$u_{k,s}(t) = A\,\mathrm{sech}(k/\nu)\mathrm{e}^{\mathrm{i}(2\beta+\gamma A^2/2)t}, \tag{9}$$

where $\nu = (2\beta)^{1/2}/(A\gamma^{1/2})$ is the soliton width, and A is a free parameter taken such that $\nu \gg 1$. In general, in the analysis of a discrete system it is useful to consider the properties of its continuous counterpart.

We mention that the continuous equation (8) has bright soliton solutions when $\beta\gamma > 0$ [1, 2]. In contrast, the discrete counterpart (1) can have bright solitons for any sign of $\beta\gamma$, provided that $\mathrm{d}^2\omega/\mathrm{d}q^2\,\gamma > 0$. This is due to the change of the dispersion sign, mentioned in section 2.1.

Another limit for an approximate solution is the anti-continuum limit [5], when $\beta = 0$. In this case, the points are decoupled from each other. Therefore, one can write a strongly localised solution with only a single excited site

$$u_k(t) = A\,\delta_{0,k}\,\mathrm{e}^{\mathrm{i}\gamma A^2 t}. \tag{10}$$

When $\beta \neq 0$, this solution is not valid, but it can be used as a basis solution in the perturbation theory or numerical simulations.

An exact profile of a DS for an arbitrary set of the system parameters can be found numerically. We look for the solution in the form $u_k(t) = U_k \exp(-i\eta t)$. Then, the soliton profile U_n and the corresponding frequency η are found from the nonlinear eigenvalue problem:

$$-\beta(U_{k+1} + U_{k-1}) - \gamma|U_k|^2 U_k = \eta U_k, \tag{11}$$

where $k = 1, \dots, N$. The derivation of this stationary equation can be checked by using symbolic calculations; see MC.2.

Eigenvalue problem (11) has N equations for $N + 1$ unknowns, namely, η and $U_1, \dots, U_N$, subject to vanishing (zero or periodic) boundary conditions for U_k. A direct numerical solution of such a problem is a difficult task. However, if we fix the value of η, then equations (11) are just a set of N nonlinear algebraic equations for N unknowns. This set can easily be solved by corresponding routines; see MC.3.

For the convergence of the numerical solution of equations (11), proper η and an initial distribution should be chosen. A value of η should be taken below the band of linear waves $\omega_{\mathrm{lin}} = [-2\beta, 2\beta]$. This can be deduced from the analysis of the dispersion relation (4); see also Problem 4 in section 3. An initial distribution can be taken in a form close to a DS, for example $U_k = A \exp(-|k|)$, where A is the soliton amplitude. An initial value of A can be taken as $A = (|\eta|/\gamma)^{1/2}$; see equation (10). However, we find that the numerical procedure converges to the on-site (inter-site) soliton, even when initially only one (two) site(s) is (are) excited. Thus, in the code, different solitons can be obtained by changing parameters η, c_1 and c_2; see MC.1 and MC.3. Having a solution for one η, one can restore the whole family of soliton solutions, by changing gradually the value of η; see also Problem 6 in section 3.

In order to solve the set of equations (11) for a given η, one can use the Newton–Raphson method, see e.g. [14], which is the default method of the Mathematica `FindRoot[]` command; see MC.3. One can write equation (11) in a vector form $\mathbf{f}(\mathbf{z}) = 0$, where $\mathbf{z} = (U_1, U_2, \dots, U_N)$, and $\mathbf{f}$ is vector of functions, $\mathbf{f} = (f_1, f_2, \dots, f_N)$. Then in the Newton–Raphson method, a root is found by iterations [14] $\mathbf{z}_{\mathrm{new}} = \mathbf{z}_{\mathrm{old}} + \delta\mathbf{z}$, where $\delta\mathbf{z}$ is a solution of a set of linear equations $\mathbf{J}(\mathbf{z}_{\mathrm{old}})\,\delta\mathbf{z} = -\mathbf{f}(\mathbf{z}_{\mathrm{old}})$, and $\mathbf{J}$ is the Jacobian matrix $J_{nm} = \partial f_n / \partial U_m$.

There are different types of DS solutions [1, 3, 5, 6, 15]. The distribution with a centre at a site is called the on-site soliton, while that with a centre between sites is called the inter-site soliton [1, 5]. Both on-site solitons and inter-site solitons can be symmetric or anti-symmetric [15]. If we define $u_k = (-1)^k v_k$, and change $\gamma \to -\gamma$ and $t \to -t$, then dynamics of $v_k(t)$ is described by the same DNLS equation. It means, in particular, that if U_k is an unstaggered mode with frequency $\eta < 0$ for $\gamma > 0$, then $(-1)^k U_k$ is a staggered mode with $\eta > 0$ for $\gamma < 0$ [16]. This is similar to plane wave profiles with $q = 0$ and $q = \pi$, respectively. We consider only unstaggered symmetric solitons for $\gamma > 0$ (but see also Problem 5 in section 3).

We can check numerically that the solution found is indeed the eigenmode. For this purpose, we integrate numerically equation (1) taking the solution of equations (11) as the initial condition; see MC.4. We expect a stationary evolution at least for small t. Such numerical simulations, using MC.4, show that on-site solitons are stable, while inter-site are unstable [1, 5]. Figure 3 shows the dynamics of stable and unstable solitons.
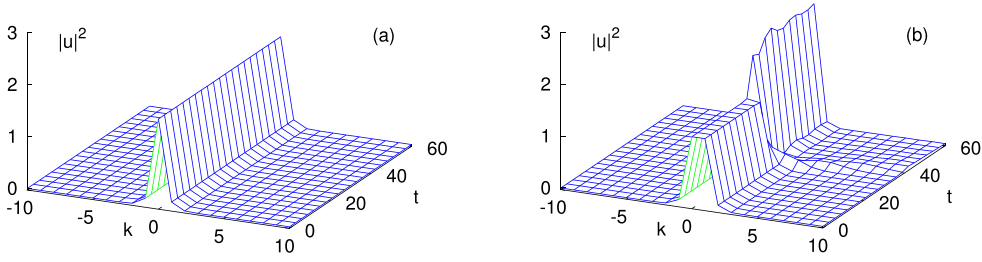
Figure 3. Dynamics of (a) on-site and (b) inter-site discrete solitons, obtained from numerical simulations of equation (1). The system parameters are $\beta = 0.5$, $\gamma = 1$, and $\eta = -2$.
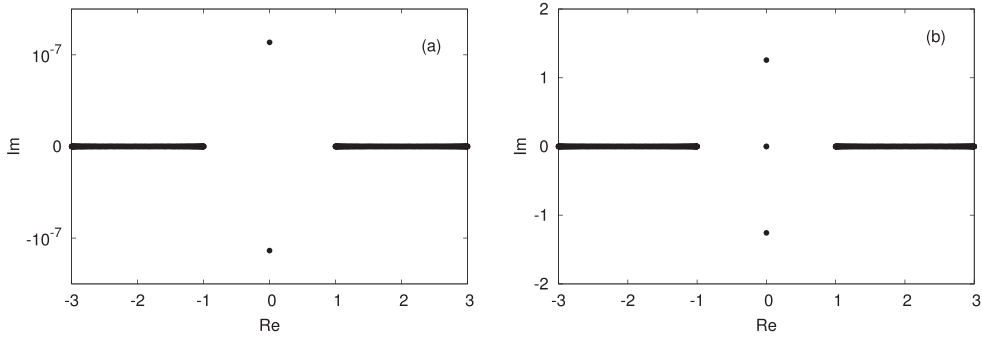

Figure 4. The distribution of modulation eigenvalues of the (a) on-site DS, and (b) inter-site DS on the complex plane of Ω. The parameters are the same as in figure 3.

Physically relevant localised waves correspond to stable solutions. Therefore, it is necessary to analyse the stability of DSs. For this purpose, similarly to equation (5), we represent the solution as the soliton with small perturbations $w_k(t)$ added:

$$u_k(t) = [U_k + w_k(t)]\exp(-\mathrm{i}\eta t). \tag{12}$$

Then, in the first approximation, the evolution of perturbations is described by the following equation:

$$\mathrm{i}\frac{\mathrm{d}w_k}{\mathrm{d}t} + \eta w_k + \beta(w_{k+1} + w_{k-1}) + \gamma U_k^2(2w_k + w_k^*) = 0, \tag{13}$$

which can be checked with the code; see MC.5. Next, we apply a procedure similar to that used for the analysis of plane waves. Namely, we obtain from equation (13) the equations for the real and imaginary parts of w_n. Then substituting these parts in the form $\sim\exp(-\mathrm{i}\Omega t)$, we get a set of linear algebraic equations in a form $\mathbf{F}\,\mathbf{A} = \Omega\mathbf{A}$, where $\mathbf{A}$ is a real vector of length $2N$ of modulation amplitudes, and $\mathbf{F}$ is the matrix of coefficients. Eigenvalues of $\mathbf{F}$ are the frequencies of modulations. A complex frequency of modulations means instability of DSs. This stability analysis is valid also for an infinite array, $N \to \infty$. All the steps described are implemented in the code; see MC.6–MC.9. The distribution of eigenvalues on the complex Ω-plane are shown in figure 4. Figure 4(a) shows that numerical accuracy of eigenvalue calculations is of order 10^{-7}.

After identification of regions of stable DSs, one can perform a further study of the discrete system. For example, one can study the evolution of arbitrary localised distributions, or the interaction of DSs.

3. Questions and problems

In this section, we suggest few problems. The aim of the problems is to help gain a deeper understanding of the methods reviewed in this paper, and also to develop skills in working with the Mathematica program in the appendix. Problems 1–7 consider the fundamental properties of plane waves and solitons. These properties are common for various systems. Solving these problems provides more insight into the nonlinear dynamics of discrete systems. Problems 8–12 and their extensions can be used as independent student projects. The references at each problem are sources for further study.

Problem 1 [**5, 17**]. Modify the program for the case of a nonlinear plane wave to obtain a plot of MI similar to figure 2.

Problem 2 [**1, 5**]. The MI theory (see equation (7)) predicts an infinite growth of perturbations for some parameters. Does such infinite growth occur in numerical simulations of the DNLS equation?

Problem 3 [**1, 5**]. Prove analytically that P and H in equation (2) are indeed the invariants of equation (1). Include to the program a calculation of P. This helps to monitor errors of numerical simulations, see MC.4.

Problem 4 [**1, 5**]. Analyse the dispersion relation (4), and figure out why the soliton frequency should satisfy $\eta < -2\beta$ ($\beta > 0$). Hint: Consider the soliton centre and tails as parts of plane waves.

Problem 5 [**4, 15, 16**]. Find anti-symmetric on-site and inter-site solitons for $\gamma > 0$. Find staggered solions for $\gamma < 0$. Analyse their stability. For convergence, modify MC.3 to excite initially 3–5 consecutive sites.

Problem 6 [**1, 5**]. Using MC.1 and MC.3 of the program, find the dependencies of the soliton amplitude and P on η.

Problem 7 [**18**]. An initial condition in a form of a localised mode (see equation (11) and MC.3) multiplied by $\exp(\mathrm{i}pk)$, where p is a constant, results in moving solitons. Modify step MC.4 of the program to generate such solitons. How does the soliton velocity depend on η and p?

Problem 8 [**19**]. Consider a linear array with a nonlinear defect, described by the following equation:

$$\mathrm{i}\frac{\mathrm{d}u_k}{\mathrm{d}t} + \beta(u_{k+1} + u_{k-1}) + \gamma\delta_{k,0}|u_k|^d u_k = 0, \tag{14}$$

where d is an arbitrary exponent. Find modes localised on the defect, and check their dynamics and stability.

Problem 9 [20]**.** Consider a semi-infinite array of waveguides, described by equation (1) with $k > 0$ and $u_0 = 0$. Find and analyse localised modes with maximum at different distances from the border ($k = 1$).

Problem 10 [21, 22]**.** Consider the cubic-quintic DNLS (C-Q DNLS) equation, which can be represented in the following form:

$$i\frac{du_k}{dt} + \beta(u_{k+1} + u_{k-1}) + \gamma|u_k|^2 u_k + \delta|u_k|^4 u_k = 0. \tag{15}$$

Modify and apply the program to find the different types of a localised solutions of C-Q DNLS equation and analyse their linear stability properties.

Problem 11 [2, 23, 24]**.** Consider the Ablowitz–Ladik (A–L) equation:

$$i\frac{du_k}{dt} + \beta(u_{k+1} + u_{k-1}) + \gamma|u_k|^2(u_{k+1} + u_{k-1}) = 0. \tag{16}$$

This equation is an integrable discretization of the NLS equation and also has some applications in physics. It has exact solitonic solutions. Modify and apply the program to analyse the localised solutions of the A–L equation. One can include the on-site cubic term to A–L equation (which destroys the integrability) and analyse the properties of localised solutions:

$$i\frac{du_k}{dt} + \beta(u_{k+1} + u_{k-1}) + \gamma|u_k|^2(u_{k+1} + u_{k-1}) + \delta|u_k|^2 u_k = 0. \tag{17}$$

This equation is called the Salerno equation.

Problem 12 [25]**.** The following inhomogeneous DNLS equation describes the nonlinear localised impurity modes:

$$i\frac{du_k}{dt} + \beta(u_{k+1} + u_{k-1}) + \gamma|u_k|^2 u_k + \kappa\delta_{k,0}|u_k|^4 u_k = 0. \tag{18}$$

Use the Mathematica code to find these modes and check their stability.

4. Conclusion

The basic steps in the study of discrete systems have been presented. These steps include the construction of plane wave solutions and soliton solutions of the discrete system, the derivation of equations for small modulations, and analysis of stability. The corresponding code is implemented in Mathematica. It is demonstrated that theoretical consideration together with numerical modelling can substantially enhance the understanding of properties of discrete systems. The approach described here is quite generic, and it can be used to study other discrete systems.

The DNLS equation considered does not include many effects. An actual distributed system can be described by more general types of discrete equations; some examples are considered in section 3. Further extension of the DNLS model includes a consideration of

two- and three-dimensional arrays. Also, one can study higher-order interactions, non-local effects, and different types of nonlinearity. One can take into account external and parametric perturbations. We believe that the method and the code presented in this work can serve as a starting point to study these systems.

Acknowledgments

Appendix. Mathematica code

Here we describe a code in Mathematica (ver.8.0) for symbolic and numerical analysis of equation (1). A supplementary file is available online at stacks.iop.org/EJP/39/055803/mmedia (both in np- and pdf-formats) provides the code with outputs, so that it gives a working example for a particular set of parameters. The code is valid for equations with nearest-neighbour terms, however it can easily be extended to more general cases. Steps MC.1-MC.4 show a finding of a localised mode and a numerical simulation of its dynamics, while MC.5–MC.10 are related to a derivation of equations for modulations and calculation of the modulation spectrum.

1. Define parameters and the DNLS equation (eqn):

```
nPoints = 200; tend = 50;
par1 = {β − >0.5, γ − >1, ω − >−2}; par2 = {c1 − >1., c2 − >1.};
eqn = I*u[k]′[t] + β*(u[k + 1][t] + u[k − 1][t])
 + γ*(u[k][t])^2*Conjugate[u[k][t]]
```

2. Derive the stationary equation (eqnStat) from the DNLS equation:

```
sub1 = u[k_] − >Function[{t}, U[k]*Exp[−I*ω*t]];
eqn1 = Simplify[eqn /. sub1, Assumptions − >{{ω, t, U[i_]} ∈ Reals}];
eqnStat = Coefficient[eqn1, Exp[−I*ω*t]]
```

3. Find eigenmode (eigenMode) from the list of stationary equations (eqnStatList):

```
eqnStatList = table[eqnStat, {k, 1, nPoints}]/.
 {U[0] − >U[nPoints], U[nPoints + 1] − >U[1]};
init = table[{U[k], (c1*KroneckerDelta[nPoints/2, k]+
 c2*KroneckerDelta[nPoints/2 + 1, k])/.par2}, {k, 1, nPoints}];
eigenMode = FindRoot[(eqnStatList /.par1)== 0, init];
init1 = table[U[k], {k, 1, nPoints}]/.eigenMode;
ListLinePlot[init1, PlotRange − >{{nPoints/2 − 10, nPoints/2
+10}, All},
 PlotMarkers − >Automatic]
```

4. Solve the DNLSE, using eigenmode as an initial condition:

```
initialCond = Table[u[k][0]== init1[[k]], {k, 1, nPoints}];
dnlsList = Join[table[(eqn/.par1)== 0, {k, 1, nPoints}]/.
 {u[0][t] − >u[nPoints][t], u[nPoints + 1][t] − >u[1][t]}, initialCond];
sol1 = NDSolve[dnlsList, Table[u[k], {k, 1, nPoints}],
 {t, 0, tend}, MaxSteps − >1000000];
fig1 = Evaluate[table[Abs[u[k][t]], {t, 0, tend}, {k, 1, nPoints}]/.sol1];
ListPlot3D[fig1, PlotRange − >All]
```

5. Derive an equation for the first correction, w[i][t]:

```
sub1 = u[k_] − >Function[{t}, (U[k] + ϵ*w[k][t])Exp[−I*ω*t]];
res1 = Coefficient[Simplify[eqn/.sub1, Assumptions − >
 {{ω, t, U[i_], ϵ} ∈ Reals, w[i_][t] ∈ Complexes}], Exp[−I*ω*t]];
eqnFirstCorr = Coefficient[Collect[res1, ϵ], ϵ, 1]
```

6. Split equation eqnFirstCorr into real and imaginary parts (wr, wi):

```
res1 = Simplify[eqnFirstCorr/.w[k_] − >Function[{t},
(wr[k][t] + I*wi[k][t])],
 Assumptions − >{wr[k_][t] ∈ Reals, wi[k_][t] ∈ Reals}];
eqnz1r = Simplify[ComplexExpand[Im[res1]]]
eqnz1i = Simplify[ComplexExpand[Re[res1]]]
```

7. Find left-hand sides (lhs1r and lhs1i) of the equations for spatial distributions:

```
sub2 = {wr[k_] − >Function[{t}, ar[k]*Exp[−I*Ω*t]],
 wi[k_] − >Function[{t}, ai[k]*Exp[−I*Ω*t]]};
eqn1r = Expand[Coefficient[Simplify[eqnz1r/.sub2], Exp[−I*Ω*t], 1]];
eqn1i = Expand[Coefficient[Simplify[eqnz1i/.sub2], Exp[−I*Ω*t], 1]];
lhs1r = Simplify[Ω*ar[k] − eqn1r/Coefficient[eqn1r, Ω*ar[k], 1]]
lhs1i = Simplify[Ω*ai[k] − eqn1i/Coefficient[eqn1i, Ω*ai[k], 1]]
```

8. Construct a matrix of coefficients in a difference form (matrF) and a vector of unknowns (vectA):

```
sub3 = {ai[0] − >ai[nPoints], ai[nPoints + 1] − >ai[1],
 ar[0] − >ar[nPoints], ar[nPoints + 1] − >ar[1]};
matrFdiff = Join[table[lhs1r, {k, 1, nPoints}]/.sub3,
 table[lhs1i, {k, 1, nPoints}]/.sub3];
vectA = Join[Table[ar[k], {k, 1, nPoints}], table[ai[k], {k, 1, nPoints}]];
```

9. Substitute eigenmode and parameters into matrFdiff, and convert it to a 'standard' form. Then find eigenvalues of matrix F:

```
m1 = matrFdiff/.eigenMode/.par1;
{bb1, matrF} = CoefficientArrays[m1, vectA];
ev1 = Eigenvalues[Normal[matrF]]; Max[Im[ev1]]
```

10. Plot eigenvalues on the complex plane:

```
ev1Fig = table[{Re[ev1[[i]]], Im[ev1[[i]]]}, {i, 1, 2*nPoints}];
ListPlot[ev1Fig, PlotRange − >All, PlotMarkers − >
 Graphics[{Blue, Thick, Circle[]}, ImageSize − >8]]
```

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